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HILL'S CASE OF THE AVERAGED PROBLEM OF THREE BODIES AND THE STABILITY OF PLANE ORBITS

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S. L. ZIGLIN and M. L. LIDOV
(Moscow)

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The Hill's case of the problem of three bodies (distance between two points is considerably smaller than the distance of their barycenter to the third point) is considered. The secular evolution is first determined using the perturbation theory by a system of equations with the Hamiltonian averaged over mean longitudes of points. The averaged problem is integrated, and analyzed for all of its admissible parameter values. Situations in which motion on plane circular orbits is unstable are revealed. In connection with this the averaged problem of three bodies is used for investigating the stability of plane circular retrograde motions for arbitrary ratios of major semiaxes of orbits. Relations between parameters of the problem for which such orbits are unstable are determined.

1. Statement of the problem. In the general problem of three bodies we denote the three points and their masses by m_0, m_1 and m_2, the radius vector of point m_k relative to m_j by $\mathbf{r}_{jk}$, and by $\mathbf{r}_2$ the radius vector of point m_2 relative to the barycenter B of points m_0 and m_1.

The limit variant of this problem, when

$$|\mathbf{r}_{01}| \ll |\mathbf{r}_2| \tag{1.1}$$

will be called the Hill case by analogy with the bounded problem.

It is assumed that the following conditions of smallness of reciprocal perturbations are satisfied:

$$\nu_1 = \frac{m_0}{m_0+m_1}\frac{|\mathbf{r}_{01}|^3}{|\mathbf{r}_2|^3} < \varepsilon, \quad \nu_2 = \frac{m_0 m_1}{(m_0+m_1)^2}\frac{|\mathbf{r}_{01}|^2}{|\mathbf{r}_2|^2} < \varepsilon, \quad \varepsilon \ll 1 \tag{1.2}$$

If the terms proportional to ν_1 and ν_2 are neglected in the equations of motion, the orbits of m_1 and m_2 relative, respectively, to m_0 and B are determined by solutions of the problem of two bodies. Here we consider the case when the "unperturbed" orbits are ellipses. Methods of the perturbation theory can be used for deriving an approximate solution when in limited time intervals $\nu_1 + \nu_2 \neq 0$.

In the absence of resonance relationships between mean motions of m_1 and m_2 the fundamental laws of orbit evolution in this problem are determined in the first approximation with respect to ε by the Hamiltonian averaged over mean anomalies (or mean longitudes) of the orbital motion of m_1 and m_2. It is said that an averaged Hamiltonian defines "secular" variation of orbit elements [1].

The doubly averaged problem of three bodies may be reduced with the use of known first integrals to the Hamiltonian problem with two degrees of freedom [1], which is, apparently, not integrable. It is important that in the limit Hill's case (1.1) of this problem one of the two angular variables is cyclic, and the problem is integrable.

The Hamilton function H of the problem of three bodies (see [1]) is conveniently represented in the form

$$H = H_1 + H_2 + H' \tag{1.3}$$

$$H_1 = \frac{1}{2\mu_1}|\mathbf{p}^{(1)}|^2 - f\frac{m_0 m_1}{|\mathbf{r}_{01}|}, \quad H_2 = \frac{1}{2\mu_2}|\mathbf{p}^{(2)}|^2 - f\frac{m_2(m_0+m_1)}{|\mathbf{r}_2|}$$

$$H' = f\frac{m_2(m_0+m_1)}{|\mathbf{r}_2|} - f\frac{m_0 m_2}{|\mathbf{r}_{02}|} - f\frac{m_1 m_2}{|\mathbf{r}_{12}|}$$

$$\mu_1 = \frac{m_0 m_1}{m_0+m_1}, \quad \mu_2 = \frac{m_2(m_0+m_1)}{m_0+m_1+m_2}$$

$$\mathbf{r}_{01} = (q_1, q_2, q_3), \quad \mathbf{r}_2 = (q_4, q_5, q_6)$$

$$\mathbf{p}^{(1)} = (p_1, p_2, p_3) = \mu_1\frac{d\mathbf{r}_{01}}{dt}, \quad \mathbf{p}^{(2)} = (p_4, p_5, p_6) = \mu_2\frac{d\mathbf{r}_2}{dt}$$

where q_i are projections on the axes of the nonrotating system of coordinates, f is the gravitational constant, and p_i and q_i are canonically conjugate variables of the problem.

Using the assumption (1.1) and the expansion of H' in series in $|\mathbf{r}_{01}|/|\mathbf{r}_2|$, we obtain the asymptotic representation

$$H' = H^\circ + \frac{\mu}{|\mathbf{r}_2|}O\left(\frac{|\mathbf{r}_{01}|^3}{|\mathbf{r}_2|^3}\right) \tag{1.4}$$

$$H^\circ = -\frac{\mu}{2}\frac{|\mathbf{r}_{01}|^2}{|\mathbf{r}_2|^3}(3\cos^2\varphi - 1), \quad \mu = f\frac{m_0 m_1 m_2}{m_0+m_1} \tag{1.5}$$

where φ is the angle between vectors $\mathbf{r}_{01}$ and $\mathbf{r}_2$. When $H' = 0$ the Hamiltonian (1.3) defines the unperturbed motions of m_1 and m_2 relative to m_0 and to the barycenter B of bodies m_0 and m_1, respectively. By neglecting the terms $O(|\mathbf{r}_{01}|^3/|\mathbf{r}_2|^3)$ we obtain the approximate variant of the perturbed problem which we shall call the Hill's case. In that case the perturbation part of the Hamiltonian is of the form (1.5).

Below for the description of evolution we use as the Keplerian osculating elements of the orbit the major semiaxis a_j, the eccentricity e_j, the inclination i_j $(j = 1, 2)$ to the basic fixed plane, as well as the uniquely connected to these canonical Delaunay elements: momenta

$$L_j = \mu_j\sqrt{fM_j}\sqrt{a_j}, \quad G_j = L_j\sqrt{1-e_j^2}, \quad \Theta_j = G_j\cos i_j$$
$$(M_1 = m_0 + m_1, \quad M_2 = m_0 + m_1 + m_2)$$

and their associated coordinates: the mean anomaly l_j, the argument of the pericenter latitude g_j and the longitude of the ascending node θ_j. In Delaunay elements

$$H_j = -\frac{\mu_j^2}{2L_j^2} \quad (j = 1, 2), \quad H' = H'(L_1, \ldots, g_1, l_1, L_2, \ldots, g_2, l_2)$$

Further simplifications are based on the following considerations [1].

1°. Except for particular (resonance) values of L_1 and L_2 there exist at fairly small ν_1 and ν_2 a 2π-periodic with respect to l_1, l_2 and other angle variables the substitution of variables

$$L_j \to \bar{L}_j, \ \ldots, \ l_j \to \bar{l}_j$$

which differs from the identical by a function of order ε. The Hamiltonian $\bar{H}$ of the problem is obtained from the input one to within terms of higher order with respect to ε in terms denoted by an upper stroke by a formal substitution of $\bar{L}_j, \ldots$ for L_j and of $\bar{l}_j$ for l_j, and its independent averaging with respect to l_1 and l_2 from 0 to 2π.

Digressing from the conditionally periodic functions of order ε that define the substitution of variables, we obtain in the first order of the perturbation theory the problem with the Hamiltonian

$$\overline{H} = -\frac{\mu_1^2}{2L_1^2} - \frac{\mu_2^2}{2L_2^2} + \overline{H}', \quad \overline{H}' = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} H' dl_1 dl_2 \tag{1.6}$$

In the averaged problem L_j are constants, since l_j are cyclic coordinates.

2°. In the considered problem the vector of kinetic moment remains unchanged. If the plane orthogonal to that vector — the so-called Laplace plane — is taken as the basic coordinate plane, the area integrals make it possible to reduce the number of degrees of freedom by two. This is formally achieved by the substitution

$$G_1 = \Gamma_1, \quad G_2 = \Gamma_2, \quad \Theta_1 = \frac{c}{2} + \frac{1}{2c}(\Gamma_1^2 - \Gamma_2^2)$$

$$\Theta_2 = \frac{c}{2} - \frac{1}{2c}(\Gamma_1^2 - \Gamma_2^2)$$

in the Hamiltonian $\overline{H'}$, where $c = \Theta_1 + \Theta_2$ is the constant of areas.

The result of this substitution is that the Hamiltonian $\overline{H'}$ of the problem now depends on four canonical variables Γ_1, Γ_2, g_1 and g_2 on parameters L_1 and L_2, and on the constant of areas c. We thus obtain a problem with two degrees of freedom.

The relative inclination of orbits I, their eccentricity e_j, and the major semiaxes a_j are determined by the following formulas:

$$\cos I = \frac{c^2 - \Gamma_1^2 - \Gamma_2^2}{2\Gamma_1\Gamma_2}, \quad e_j = \sqrt{1 - \frac{\Gamma_j^2}{L_j^2}}, \quad a_j = \frac{L_j^2}{\mu_j^2 f M_j} \quad (j = 1, 2) \tag{1.7}$$

3°. If the perturbing Hamiltonian H' is restricted to the Hill approximation H° (1.5), the variable g_2 vanishes from the averaged Hamiltonian $\overline{H^\circ}$, and the considered problem proves to be integrable. The explicit formula for $\overline{H^\circ}$ is

$$\overline{H}^\circ = -\frac{\mu a_1^2}{8a_2^3(1-e_2^2)^{3/2}}[3(1-e_1^2)(1+\cos^2 I) + 15(\cos^2 g_1 + \cos^2 I \sin^2 g_1) - 6e_1^2 - 4] \tag{1.8}$$

where $\cos I$ and e_j are determined by (1.7).

2. Results of the qualitative analysis. Since $\overline{H^\circ}$ is independent of g_2, hence $\Gamma_2 = \text{const}$ is the first integral of the system. As a corollary we find that the eccentricity e_2 of the external body orbit remains unchanged in the process of evolution. When $\overline{H^\circ}$ is constant, formula (1.8) determines the integral curve in the plane Γ_1, g_1. We introduce below the notation $g_1 = \omega$ more usual for Keplerian elements, and define integral curves in the plane ε, ω, where

$$\varepsilon = 1 - e_1^2 = \Gamma_1^2 / L_1^2$$

The topology of the set of integral curves in that plane depends on two parameters α and β

$$\alpha = c/L_1, \quad \beta = \Gamma_2/L_1 \tag{2.1}$$

The reasonable restrictions

$$-1 \leqslant \cos I = \frac{\alpha^2 - \beta^2 - \varepsilon}{2\beta\sqrt{\varepsilon}} \leqslant 1, \tag{2.2}$$

$$0 \leqslant \varepsilon \leqslant 1$$

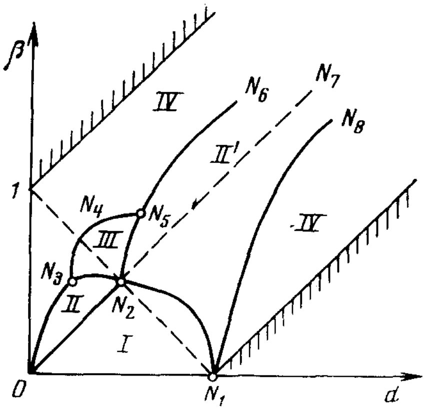

Fig. 1

isolate set M of admissible values of parameters α and β

$$M = \{(\alpha,\ \beta) \in R^2,\quad |\alpha - \beta| \leqslant 1,\quad \alpha \geqslant 0,\ \beta \geqslant 0\}$$

For any point $(\alpha, \beta) \in M$ it is possible to indicate such planet masses and orbits for which parameters α and β have specified values, and the restrictions (1.1) and (1.2) imposed for the derivation of (1.8) are satisfied.

The above analysis shows that the set M is divided by the inequalities specified below into the following nonintersecting regions:

Region I: $\beta < \alpha$ and $3\beta^2 + \alpha^2 < 1$

Region II: $\alpha + \beta < 1$ and $0 < (\beta - \alpha)(\alpha + \beta)^2 < 5\alpha\,[1 - (\alpha + \beta)^2]$

Region II′: $\alpha + \beta > 1$ and $0 < 2\,(3\beta^2 + \alpha^2 - 1) < 5\,[4\beta^2 - (\alpha^2 - \beta^2 - 1)^2]$

Region III: $(\alpha - \beta)^2 < \frac{2}{3}\left(\frac{\beta^2}{2} + \alpha^2 + \frac{5}{8}\right) < \min\{1,\ (\alpha + \beta)^2\}$ and $(\alpha^2 - \beta^2)^2 < \left(\frac{32}{135}\left(\frac{\beta^2}{2} + \alpha^2 + \frac{5}{8}\right)\right)^3$ and $5\alpha\,[1 - (\alpha + \beta)^2] < (\beta - \alpha)(\alpha + \beta)^2$, if $\alpha + \beta < 1$, $5\,[4\beta^2 - (\alpha^2 - \beta^2 - 1)^2] < 2\,(3\beta^2 + \alpha^2 - 1)$, if $\alpha + \beta > 1$

Region IV complements the above regions to complete set M.

Set M and its subdivision are represented in Fig. 1.

The qualitative behavior of integral curves in regions I, II and II′, III and IV are presented in Figs. 2, a, 2, b, 2, c and 2, d, respectively.

Results of the analysis that correspond to boundary lines in Fig. 1 are shown in Fig. 3, as follows: (a) — N_1N_2; (b) — $N_3N_2N_5$; (c) — $N_3N_4N_5$, (d) — ON_3, N_5N_6, N_1N_8, (e) — ON_2, (f) — N_2N_7. Owing to the problem symmetry with respect to ω only segment $0 \leqslant \omega \leqslant \pi/2$ is shown.

The admissible range of values of ε depends on α and β, and is determined by the inequalities $\varepsilon_{\min} = (\alpha - \beta)^2 \leqslant \varepsilon \leqslant \varepsilon_{\max} = \min\{1,\ (\alpha + \beta)^2\}$. The coordinates of singular points are determined by formulas

$$A{:}\ (\omega = 0,\ \varepsilon = 3\beta^2 + \alpha^2),\quad B{:}\ (\omega = \pi/2,\ \varepsilon = \varepsilon_B)$$

$$E{:}\left(\begin{array}{l} \omega = \omega_1,\ \varepsilon = (\alpha + \beta)^2 \ \text{ for } \ \alpha + \beta \leqslant 1 \\ \omega = \omega_2,\ \varepsilon = 1 \ \text{ for } \ \alpha + \beta > 1 \end{array}\right)$$

$$A'{:}\ (\omega = \pi/2,\ \varepsilon = \varepsilon_{A'})$$

where

$$\omega_1 = \arcsin\sqrt{\frac{(\beta - \alpha)(\alpha + \beta)^2}{5\alpha\,[1 - (\alpha + \beta)^2]}}$$

$$\omega_2 = \arcsin\sqrt{\frac{2\,(3\beta^2 + \alpha^2 - 1)}{5\,[4\beta^2 - (\alpha^2 - \beta^2 - 1)^2]}}$$

and ε_B and $\varepsilon_{A'}$ are roots of equation

$$\varepsilon^3 - \left(\frac{\beta^2}{2} + \alpha^2 + \frac{5}{8}\right)\varepsilon^2 + \frac{5}{8}(\alpha^2 - \beta^2)^2 = 0$$

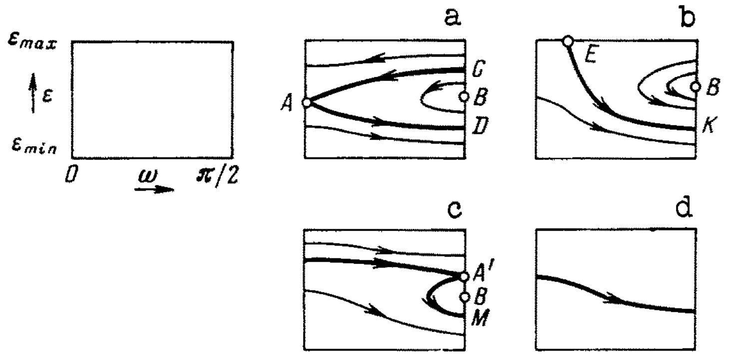

Fig. 2

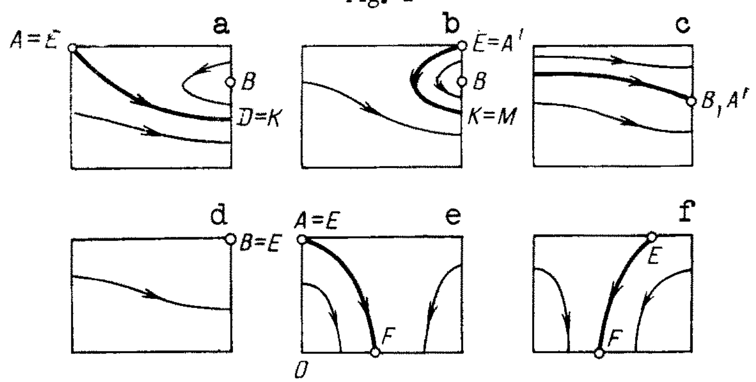

Fig. 3

Let us note some of the qualitative laws that follow from these results.

1°. There exist configurations determined by the equilibrium point B that are in a particular sense stable and steady. Semiaxes, eccentricities of both orbits, relative inclination of their planes, and the position of the inner orbit pericenter relative to the Laplace plane are retained in related solutions. If the evolution of the position of the external orbit pericenter is neglected, it is possible to say that the orbits evolve as a solid body.

2°. A circular solution $e_1 = 0$ ($\varepsilon = 1$) always exists when $\alpha + \beta \geqslant 1$. It will be seen from Fig. 2 that that solution in regions I, III and IV is stable, while in region II' it is unstable. With the use of formulas which define the geometry of region II' it is possible to prove the following statement. For the circular orbits to be unstable it is necessary and sufficient that the relative inclination I satisfies the inequalities

$$\cos I^2 \leqslant \cos I \leqslant \cos I^3, \quad \text{if} \quad \beta \geqslant {}^1/_2$$

$$\cos I^1 \leqslant \cos I \leqslant \cos I^3, \quad \text{if} \quad \beta \leqslant {}^1/_2$$

$$\cos I^1 = -2\beta, \quad \cos I^{2,3} = -\frac{1}{10\beta} \pm \sqrt{\frac{3}{5} + \frac{1}{100\beta^2}}$$

In the considered problem I^k $(k = 1, 2, 3)$ represent the critical inclinations. At transition through the critical inclination the orbit stability is disturbed with respect to eccentricity e_1 and relative inclination I.

3°. Solution

$$\cos I = \frac{\alpha^2 - \beta^2 - \varepsilon}{2\beta\sqrt{\varepsilon}} = \pm 1 \tag{2.3}$$

corresponds to the plane case. When the kinetic moments of orbital motion have the same direction, then $\cos I = 1$. When points m_1 and m_2 move in opposite directions, $\cos I = -1$ (the so-called retrograde motion).

It follows from (2.3) that in the plane case the eccentricity e_1 does not vary, and the stability (instability) of solution $|\cos I| = 1$ with fixed α and β occurs simultaneously with the stability (instability) of solution $\varepsilon = \varepsilon_{min}$ or $\varepsilon = \varepsilon_{max}$. Formula (2.3) and Fig. 2 imply that solution $\cos I = 1$ is always stable, while solution $\cos I = -1$ is unstable only then when the parameters of the problem belong to region II.

A clearer interpretation can be obtained by specifying the condition of belonging to region II for every β and $\cos I = -1$ as the condition for the orbit eccentricity

$$e^1(\beta) \leqslant e_1 \leqslant e^2(\beta)$$

where $e^i(\beta)$ are determined by (2.3) and the equations of region II boundary. The result can then be formulated as follows. A range of values of the orbit eccentricity e_1 for which the retrograde motion is unstable exists for any β for which there are points of region II. The values $e^i(\beta)$ are critical values of the eccentricity.

4°. It follows from Figs. 3, e and f that, independent of initial conditions, for $\alpha = \beta$ at the end of evolution $\varepsilon = \varepsilon_{min} = 0$, i.e. the orbit eccentricity e_1 increases up to unity. Since the orbit semiaxis a_1 remains unchanged, the orbit pericenter vanishes for $e_1 = 1$, and a collision between bodies m_0 and m_1 takes place. Formula (2.2) implies that the initial inclination I can have any value in the range of 90° to 180°, but $I \to 90°$ always when $e_1 \to 1$.

We separately note the case of $\alpha = \beta = 1/2$. Boundaries of the four regions I, II, II' and III converge at that point, and there exist plane circilar orbits ($\cos I = -1$, $e_1 = 0$). Since the point borders on region II, the plane orbits are unstable. Since it also belongs to line $\alpha = \beta$, hence during the evolution the plane circular orbit is, first, transformed into a strongly elongated ellipse with an inclination close to 90° and, then, bodies m_0 and m_1 collide. That case is considered below in a numerical example.

3. Orbits of small eccentricities and relative inclination. The assumption (1.1) which defined the Hill case made it possible to investigate the averaged problem for arbitrary eccentricities and relative inclinations, in the course of which a multiplicity of values of the problem parameters for which the plane (retrograde) circular motion is unstable. The stability of plane circular orbits can be investigated in the averaged problem without the restriction (1.1) on which the previous analysis was based.

Let us take $\bar{H}'$ (1.6) as the input Hamiltonian. We know that the averaged problem admits the circular solution $e_1 = e_2 = 0$, to which in the Delaunay variables correspond the equalities $\Gamma_1 = L_1$ and $\Gamma_2 = L_2$.

The analysis is conveniently carried out in the new canonical variables

$$p_j = \sqrt{2(L_j - \Gamma_j)}\cos g_j, \quad q_j = -\sqrt{2(L_j - \Gamma_j)}\sin g_j \quad (j = 1, 2)$$

Since $p_j = q_j = 0$ $(j = 1, 2)$ must be the solution of the Hamilton equations, hence the expansion of $\overline{H}'$ in a series in p_j and q_j begins with the quadratic part of H_2. It can be shown that the latter can be represented in the form

$$H_2(p, q) = \frac{\beta_1}{2}(p_1^2 + q_1^2) + \frac{\beta_2}{2}(p_2^2 + q_2^2) + \gamma(p_1p_2 - q_1q_2) + \frac{\delta_1}{2}p_1^2 + \frac{\delta_2}{2}p_2^2 + \delta_3 p_1 p_2$$

where β_i, γ and δ_i are coefficients that depend on parameters c, L_1 and L_2, and on the mass of planets. The characterisitc equation of the related linear system is of the form

$$\lambda^4 + \lambda^2[\beta_1(\beta_1 + \delta_1) + \beta_2(\beta_2 + \delta_2) - 2\gamma(\gamma + \delta_3)] + (\beta_1\beta_2 - \gamma^2)[(\beta_1 + \delta_1)(\beta_2 + \delta_2) - (\gamma + \delta_3)^2] = 0 \quad (3.1)$$

Owing to the complex dependence of coefficients of this equation on parameters, its roots were not analyzed in the general case. We shall solve the considered problem of stability of plane circular orbits with respect to eccentricities with the use of a special method.

Below we consider orbits that are close to plane retrograde ones $(\cos I \approx -1)$. We substitute for the area constant c the quantity $\delta = c - c_0$, where $c_0 = c_0(L_1, L_2)$ is the constant of areas which corresponds to plane circular orbits with $\cos I = -1$, i.e. $c_0^2 = (L_1 - L_2)^2$. It is not difficult to see that for circular orbits $\delta \geqslant 0$.

It can be shown that the asymptotic representation

$$\beta_1 = \beta_{10} + \delta_4, \quad \beta_2 = \beta_{20} + \delta_5, \quad \gamma = \gamma_0 + \delta_6 \quad (3.2)$$

$$\beta_{10} = -fm_2\left(\frac{1}{2L_1} - \frac{1}{4L_2}\right)B_1', \quad \beta_{20} = -fm_2\left(\frac{1}{2L_2} - \frac{1}{4L_1}\right)B_1'$$

$$\gamma_0 = -fm_2\frac{B_2'}{4\sqrt{L_1L_2}}$$

$$B_k' = m_0B_k\left(\frac{m_1}{m_0 + m_1}a_1, a_2\right) + m_1B_k\left(\frac{m_0}{m_0 + m_1}a_1, a_2\right)$$

$$B_k(x_1, x_2) = \frac{2}{\pi}\int_0^{\pi}\frac{x_1x_2\cos k\varphi\, d\varphi}{(x_1^2 + x_2^2 - 2x_1x_2\cos\varphi)^{3/2}} \quad (k = 1, 2)$$

where a_i are orbit semiaxes and $\delta_1, \delta_2, \ldots, \delta_6$ tend to zero together with δ is valid.

Let us first investigate the case of $\delta = 0$. We denote the expression for H_2 when $\delta = 0$ by H_{20}. The condition of fixed sign for H_{20} is of the form

$$\beta_{10}\beta_{20} - \gamma_0^2 > 0 \quad (3.3)$$

Thus, according to the Liapunov theorem, the plane circular (retrograde) motion is stable with respect to eccentricities and relative inclinations, also, for fairly small $\delta > 0$.

Further analysis is based on the validity of the inequality $(\beta_{10} + \beta_{20})^2 - 4\gamma_0^2 > 0$ which follows from inequalities $B_1 > B_2 > 0$ [1]. It can be shown by using this inequality in the analysis of roots of (3.1) that the following statement is valid. When

$\delta = 0$ and $\beta_{10}\beta_{20} - \gamma_0^2 \leqslant 0$ the roots of Eq. (3. 1) are pure imaginary and different, except when $\beta_{10}\beta_{20} - \gamma_0^2 = 0$ (a pair of zero roots).

Thus when $\beta_{10}\beta_{20} - \gamma_0^2 < 0$ the plane circular solutions are stable in a linear approximation and, since the roots are different, the stability is strong, i. e. in the linear approximation stability exists for fairly small δ.

It can be shown that zero roots do not generate instability of the system defined by the Hamiltonian H_{20}, i. e. the circular plane (retrograde) motion is stable in linear approximation with respect to perturbations which do not alter the constant $\delta = 0$.

However, as will be now shown, for small $\delta > 0$ in the neighborhood of surface Π_0: $\beta_{10}\beta_{20} - \gamma_0^2 = 0$ there exists in the parameter space of the problem a complete region (of width $\sim \delta$) in which the circular solution is unstable. In fact, it follows from Eq. (3. 1) that when

$$(\beta_1\beta_2 - \gamma^2)^2 + (\beta_1\beta_2 - \gamma^2)X < 0 \qquad (3.4)$$
$$X = \beta_1\delta_2 + \beta_2\delta_1 + \delta_1\delta_2 - 2\gamma\delta_3 - \delta_3^2$$

the characteristic equation has a positive real root, and the solution of the system with Hamiltonian H_2 is unstable with respect to the eccentricity,

For fixed δ the quantities β_i, γ, δ_i depend on the set of parameters $z = (L_1, L_2, m_0, m_1$ and $m_2)$. Let us consider in space z the hypersurface Π: $\beta_1\beta_2 - \gamma^2 = 0$ which tends to Π_0 when $\delta \to 0$. When passing through that surface $(\beta_1\beta_2 - \gamma^2)$ changes its sign. If function X does not vanish at surface Π , then to one side of that surface there exists a set of values of the problem parameters for which the circular solution is unstable with respect to eccentricities, in particular, when the semiaxes a_1 and a_2 are varied.

It remains to show that the condition $\beta_1\beta_2 - \gamma^2 = 0$ does not imply that $X = 0$. Owing to the analytic dependence of X on parameters z , it is sufficient to prove this in some limit case. When $a_1/a_2 \to 0$ and $\delta \to 0$, we can obtain the following asymptotic estimates

$$\beta_1 = -\frac{3}{4}A\left(\frac{2}{L_1} - \frac{1}{L_2} + O_1(\delta) + O_2\left(\frac{a_1}{a_2}\right)\right)$$
$$\beta_2 = -\frac{3}{4}A\left(\frac{2}{L_2} - \frac{1}{L_1} + O_3(\delta) + O_4\left(\frac{a_1}{a_2}\right)\right)$$
$$\gamma = AO_5\left(\frac{a_1}{a_2}\right), \quad \delta_1 = -\frac{15}{2}A\frac{\delta}{L_1^2L_2}\left(|L_1 - L_2| + O_6(\delta) + O_7\left(\frac{a_1}{a_2}\right)\right)$$
$$\delta_2 = AO_8\left(\frac{a_1}{a_2}\delta\right), \quad \delta_3 = AO_9\left(\frac{a_1}{a_2}\delta\right), \quad A = \mu\frac{a_1^2}{a_2^3}$$

It is evidently possible to find $O_{10}(\delta)$ and $O_{11}(a_1/a_2)$ such that the relationship $\beta_1\beta_2 - \gamma^2 = 0$ is satisfied for

$$L_1 = 2L_2 + O_{10}(\delta) + O_{11}(a_1/a_2)$$

and the calculation by formula (3.4) yields

$$X = \frac{135}{32}\mu^2\frac{a_1^4}{a_2^6}\frac{\delta}{L_2^3}\left[1 + O_{12}(\delta) + O_{13}\left(\frac{a_1}{a_2}\right)\right] \neq 0$$

This proves the existence of a region of unstable circular orbits close to the surface $\beta_{10}\beta_{20} - \gamma_0^2 = 0$. Using formula (3. 2) for β_{10}, β_{20} and γ_0, it is possible to solve this equation for L_2

$$L_2 = \frac{L_1}{4}\left(5 - \frac{B_1'^2}{B_1'^2} \pm \sqrt{\left(5 - \frac{B_2'^2}{B_1'^2}\right)^2 - 16}\right) \tag{3.5}$$

This implies that for any values of a_1, a_2, m_0 and m_1 and a fairly small $\delta > 0$ there exist two ranges of values of m_2 for which the related circular value is unstable. Length of these ranges tends to zero when $\delta \to 0$.

Note. 1°. In the Hill's case formula (3.5) yields two solutions

a) $L_1 = 2L_2$ $(\alpha = c_0 / L_1 = {}^1/_2,\ \beta = L_2 / L_1 = {}^1/_2)$

b) $L_2 = 2L_1$ $(\alpha = 1,\ \beta = 2)$

It was shown above that in the case (a) $X \neq 0$, and the conclusion about the instability of plane (retrograde) circular orbits when $\alpha = \beta = {}^1/_2$.

In case (b) the described procedure does not allow the proof of the inequality $X \neq 0$ with an accuracy to within the Hill approximation. This also agrees with that within the Hill approximation no instability was revealed for $\alpha = 1$ and $\beta = 2$. This does not, however, exclude the appearance of instability in the next approximation with respect to a_1 / a_2. That case requires a more detailed analysis.

2°. The complete (nonlinear) analysis of the problem of stability of circular plane (retrograde) orbits requires, furthermore, the exposure in region $\beta_1\beta_2 - \gamma^2 < 0$ of surfaces on which resonance relationships (of third and fourth order) between the roots of the characteristic equation (3.1) make their appearance, and to analyze the system for stability with allowance for terms of the third and fourth order with respect to p_j and q_j in the Hamiltonian $\overline{H}'$.

The considered system has always two degrees of freedom, hence outside the indicated resonance surfaces a strict (nonlinear) stability with respect to eccentricities and inclination is generally present.

4. Comparison with the results of numerical integration. Conclusions of the approximate analysis of the averaged system were checked by the method of numerical integration in the following formulation.

Rigorous equations of motion of three points gravitating in conformity with the law of universal gravitation were considered in a system of nonrevolving Cartesian coordinates with origin in m_0

Dimensionless quantities of length, mass, and time determined so as to have the gravitational constant equal to unity were used.

The following values of the mass of planets and initial elements of orbits were selected:

$$m_0 = m_1 = 1, \quad m_2 = {}^1/_9 \tag{4.1}$$
$$a_1 = 1, \quad e_1 = 0.08, \quad i_1 = 175.4°, \quad \omega = g_1 = 26.6°, \quad \theta_1 = 180°, \quad \vartheta_1 = 0$$
$$a_2 = 5.3, \quad e_2 = 0, \quad i_2 = 0, \quad g_2 = 0, \quad \theta_2 = 0, \quad \vartheta_2 = 0$$

where ϑ_i is the true anomaly, and subscripts *1* and *2* relate, respectively, to the orbit of m_1 relative to m_0, and to orbit of m_2 relative to the barycenter of m_0 and m_1. The Cartesian coordinates and components of velocity of m_1 and m_2 relative to m_0 at the instant of time $t = 0$ were computed for the data in (4.1) using the formulas of the problem of two bodies.

The input data were chosen so as to correspond to the most interesting case when $\alpha \approx \beta \approx {}^1/_2$. These conditions do not uniquely determine the mass of points and the orbit semiaxes. The remaining arbitrariness was used in the form of a compromise for obtaining with a reasonably small a_1/a_2 (the Hill approximation) for a not too small ν_1 (1. 2). An excessively small ν_1 would have resulted in an increase of the time during which significant effects of evolution become apparent.

For $\alpha = \beta = 1/2$ the first of formulas (2. 2) implies the following dependence of the orbit eccentricity on the inclination I of orbits:

$$e_1 = \sin I \tag{4. 2}$$

The input values of e_1 and I satisfy this relation. To check the revealed instability of plane circular orbits it is necessary to select e_1 (and $\sin I$) closer to zero. It is, however, possible to show that with small initial e_1 the evolution is extremely slow. A twofold increase of eccentricity occurs in a time interval of order $1/e_1^2$ (the depeendence of the computer working time is similar). In this sense $e_1 = 0.08$ is a compromise.

The curve of function (4. 2) is shown in Fig. 4, where the dots denote the osculating values of e_1 and I determined by numerical integration. The time corresponding to points 1 — 7 is given below

N	1	2	3	4	5	6	7
$t \cdot 10^{-6}$	0	1	1.1	1.128	1.136	1.14	1.1 424

The numerical integration confirms the results of the present analysis not only qualitatively, but also quantitatively with a reasonable degree of accuracy.

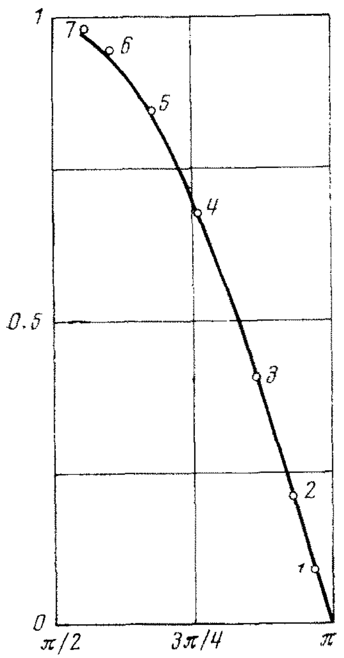

Fig. 4

Certain discrepancy between computed points and the theoretical curve for $e_1 \approx 0.96$ and $I \approx 107.6°$ can be explained by the difference of the osculating elements from the average.

5. Historical notes. The averaged problem of three bodies was considered in [3] in the Hill approximation. The case of $\beta \gg 1$ which in fact is equivalent to the limited problem of three bodies, was analyzed. It was shown that the problem is integrable. Only the case of small eccentricities and inclinations, when parameters α and β belong to region IV and the phase pattern in the plane (ε, ω) is of the simplest kind (Fig. 2, d), were qualitatively investigated.

The limited problem was fully investigated in [4, 5] where new kinds of motion that correspond to region II' in the plane (α, β) were qualitatively disclosed, the effect of collapse on the central body of orbits with a 90° initial inclination (a particular case of condition $\alpha = \beta$ when $m_1 \to 0$) evinced, and the critical inclination $\cos^2 I = 3/5$ determined.

The unlimited problem in the Hill approximation was considered in [6 — 10]. It was shown in [6] that the unlimited problem is integrable, while its qualitative analysis was carried out in [6 — 10] for fairly large β. No cases were indicated in which (α, β) belong to regions I, II and III shown in Fig.1, where significant new effects that are due to the problem unrestrictedness become apparent.

Formulas for critical inclinations $I^{1,2,3}$ were obtained in [11] by a different method. Stability of straight motions at small inclinations and eccentricities is known from the analysis of Lagrange secular perturbations (see, e. g. [1]).

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